Stat 274
Theory of Interest

Lecture 2: Equations of Value and Yield Rates

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Equations of Value

When using compound interest with a single deposit of c at time 0, the equation of value is

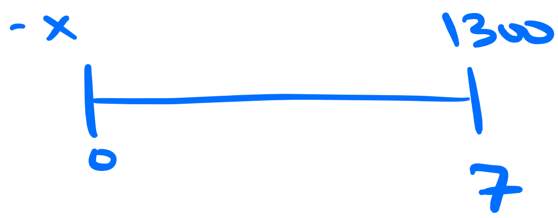
$$A_c(t) = c(1+i)^t$$

Examples

1. You win a prize and then invest that prize at 3% annually compounding interest. In 7 years, you have 1300 in the account. How much was the prize worth? [1057.02]
2. You invest 1000 at 2% annually compounding interest. You now have 1200. How long was the money in the account? [9.207]
3. You invest 100 for five years and end up with 110. What interest rate (annually compounding) did you earn? [1.924%]
4. At 8% annually compounding interest, how long will it take to double your money? [9.006]

Examples

You win a prize and then invest that prize at 3% annually compounding interest. In 7 years, you have 1300 in the account. How much was the prize worth? [1057.02]



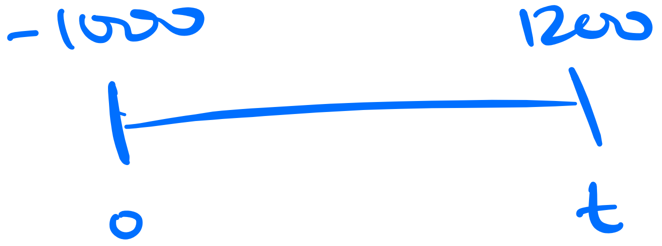
$$x\,(1+0.03)^7 = 1300$$

$$x = 1300\,v^7$$

$$x = \frac{1300}{(1.03)^7}$$

Examples

You invest 1000 at 2% annually compounding interest. You now have 1200. How long was the money in the account? [9.207]



$$1000\,(1.02)^t = 1200$$

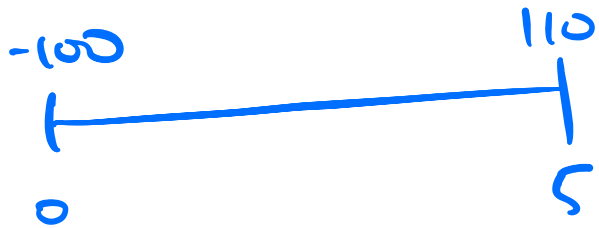
$$1.02^t = \frac{1200}{1000}$$

$$t \log(1.02) = \log(1.2)$$

$$t = \frac{\log(1.2)}{\log(1.02)}$$

Examples

You invest 100 for five years and end up with 110. What interest rate (annually compounding) did you earn? [1.924%]



$$100(1+i)^5 = 110$$

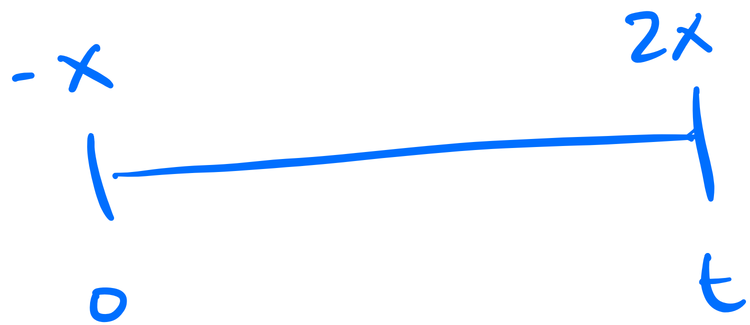
$$(1+i)^5 = \frac{110}{100}$$

$$1+i = \left(\frac{110}{100}\right)^{1/5}$$

$$i = \left(\frac{110}{100}\right)^{1/5} - 1$$

Examples

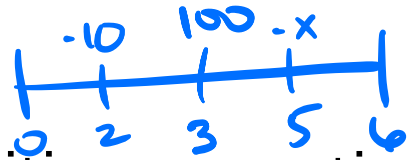
At 8% annually compounding interest, how long will it take to double your money? [9.006]



$$x(1.08)^t = 2x$$

$$1.08^t = 2$$

$$t = \frac{\log(2)}{\log(1.08)}$$



$\frac{100}{(1+i)^3}(1+i)^6 = 100(1+i)^3$

Multiple investments (can be positive or negative), each at time t_k, will have a time τ value of B at time T according to this equation,

$$\sum_k C_{t_k} \frac{a(\tau)}{a(t_k)} = B \frac{a(\tau)}{a(T)} \qquad \text{(time } \tau \text{ equation of value)}$$

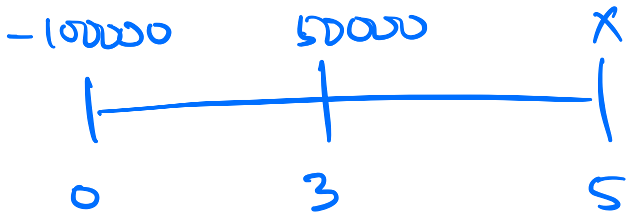
With the following special cases,

$$\sum_k C_{t_k} \frac{a(T)}{a(t_k)} = B \qquad \text{(time T equation of value)}$$

$$\sum_k C_{t_k} v(t_k) = Bv(T) \qquad \text{(time 0 equation of value)}$$

100000

$i = 0.04$



@ $t = 2$

$$100000(1.04)^2 = \frac{50000}{1.04} + \frac{X}{1.04^3}$$

@ $t = 3$

$$100000(1.04)^3 = 50000 + \frac{X}{1.04^2}$$

@ $t = 400$

$$100000(1.04)^{400} = 50000(1.04)^{397} + X(1.04)^{395}$$

Examples

1. You borrow 1000 at an annual rate of 10% (note that for the rest of the class we will assume annual compound interest unless otherwise stated) and pay it back with one payment of 600 at the end of the first year and another payment of X at the end of the second. Calculate X. [550]
2. A loan can be paid off by two different payment streams. The first is 100 at times 5 and 10 and another 200 at time 15. The second is a single payment of 400 at time t. Interest is 4.5%. Calculate t. [10.86193]
3. You deposit 500 at time 0, another 300 at time 1, and a final 200 at time 2. Immediately after your last deposit, your account has 1100. Calculate the interest rate. [0.07477]
4. You deposit 500 at time 0, another 300 at time 1, and a final 200 at time 2. At time 3, your account has 1100. Calculate the interest rate. [4.2%]

Examples

You borrow 1000 at an annual rate of 10% (note that for the rest of the class we will assume annual compound interest unless otherwise stated) and pay it back with one payment of 600 at the end of the first year and another payment of X at the end of the second. Calculate X. [550]

$1000(1.1) = 1100$

$500(1.1) = 550$

@ $t=2$

$$1000(1.1)^2 = 600(1.1) + x$$

@ $t=0$

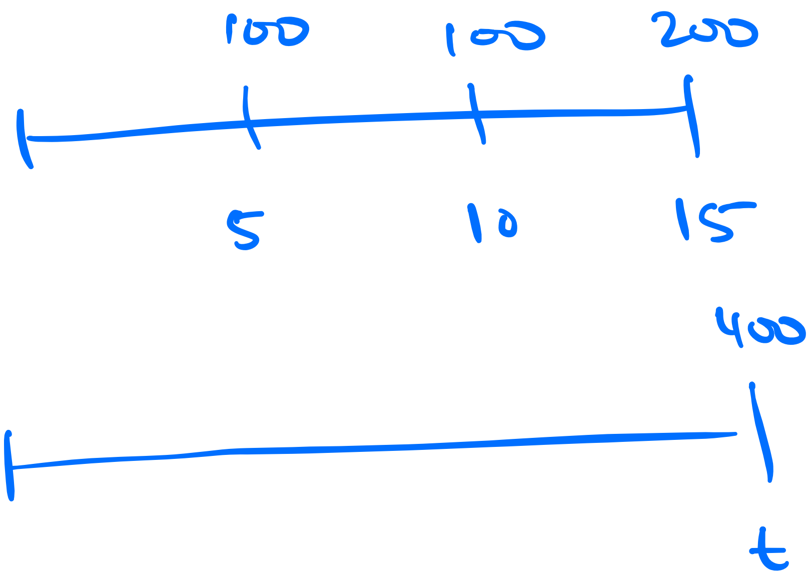
$$1000 = \frac{600}{1.1} + \frac{x}{1.1^2}$$

@ $t=74$

$$1000(1.1)^{74} = 600(1.1)^{73} + x(1.1)^{72}$$

Examples

A loan can be paid off by two different payment streams. The first is 100 at times 5 and 10 and another 200 at time 15. The second is a single payment of 400 at time t. Interest is 4.5%. Calculate t. [10.86193]



@ t=0

$$\frac{100}{1.045^5} + \frac{100}{1.045^{10}} + \frac{200}{1.045^{15}}$$

$$\frac{400}{1.045^t} = 400(1.045)^{-t}$$

Examples

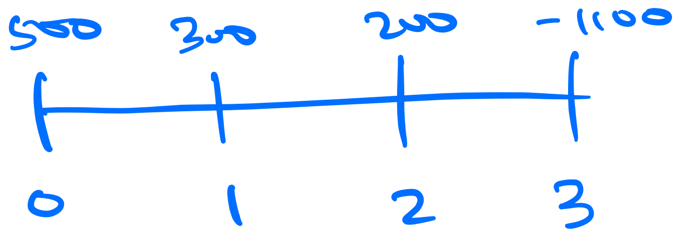
You deposit 500 at time 0, another 300 at time 1, and a final 200 at time 2. Immediately after your last deposit, your account has 1100. Calculate the interest rate. [0.07477]

@ $t=2$

$$500(1+i)^2 + 300(1+i) + 200 = 1100$$

$$500x^2 + 300x - 900 = 0$$

You deposit 500 at time 0, another 300 at time 1, and a final 200 at time 2. At time 3, your account has 1100. Calculate the interest rate. [4.2%]

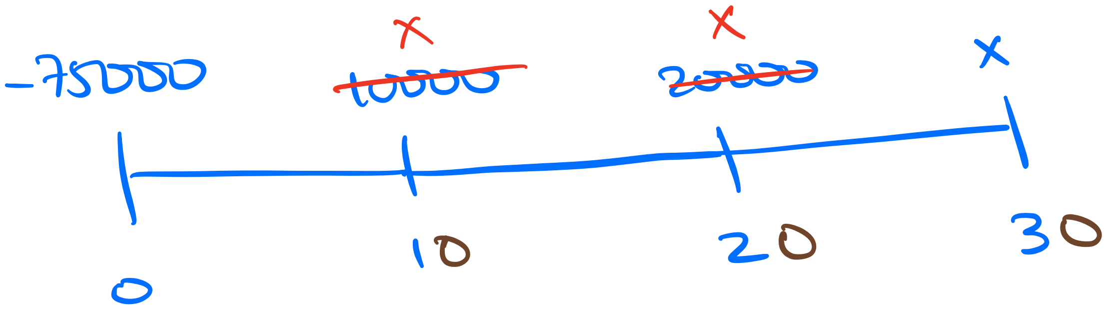


@ $t=3$

$$500(1+i)^3+300(1+i)^2+200(1+i)-1100$$

75000

$i = 0.05$



@ $t=3$ $\quad 75000(1.05)^{30} = \overset{x}{\cancel{10000}}(1.05)^{20} + \overset{x}{\cancel{20000}}(1.05)^{10} + x$

$$75000(1.05)^3 = x(1.05^2 + 1.05 + 1)$$

@ $t=0$ $\quad 75000 = \overset{x}{\cancel{10000}}(1.05)^{-10} + \overset{x}{\cancel{20000}}(1.05)^{-20} + x(1.05)^{-30}$

$$75000 = x\left(\frac{1}{1.05} + \frac{1}{1.05^2} + \frac{1}{1.05^3}\right)$$

Yield Rates

When calculating unknown interest rates, we are finding the yield rate for the investment.

As long as the sign of the payments only changes once, we are guaranteed a unique interest rate $i > -1$.

Reinvestment

B: 100 -20 -20 -20 -20 -20 -20

A: -100 x

0 1 2 2 4 5 6

$$100 = x(1+i)^{-6}$$

$$100(1+i)^6 = x$$

Sometimes the yield rate of the borrower and the lender will be different.

$$100(1+i)^6 = 126.16$$

Example: Abby pays Brandon 100, in return he pays 20 at times 1, 2, 3, 4, 5, and 6. Each time Abby gets a payment, she invests it at 2% interest and then withdraws all the money at the end of the 6 years. Calculate the interest rate paid by Brandon and Abby's yield rate. [A = 0.03949, B = 0.0547179]

$$x = 20(1.02)^5 + 20(1.02)^4 + 20(1.02)^3 + 20(1.02)^2 + 20(1.02) + 20$$

$$= 126.16$$

$$i = \left(\frac{126.16}{100}\right)^{1/6} - 1$$

$$\left(\frac{100}{x}\right)^{-1/6} - 1 = i_A$$

SOA Examples

B: $100\left(1+\frac{i^{(2)}}{2}\right)^{2\cdot 7.25}$ P: $100 e^{7.25\delta}$ $100 e^{\int_0^{7.25}\delta\, dt}$

Bruce deposits 100 into a bank account. His account is credited interest at a nominal rate of interest of 4% convertible semiannually.

At the same time, Peter deposits 100 into a different account. Peter's account is credited at a force of interest of δ.

After 7.25 years, the value of each account is the same.

Calculate δ. [0.0396]

0.0388, 0.0392, 0.0396, 0.0404, 0.0414 (FM is a multiple choice test and these are the options)

$\cancel{100}\left(1+\frac{i^{(2)}}{2}\right)^{\cancel{14.5}} = \cancel{100}\, e^{\cancel{7.25}\delta}$

$\left(1+\frac{0.04}{2}\right)^2 = e^{\delta}$

$\delta = \log\left(1.02^2\right)$

$E: 100\left(1+\frac{q}{2}\right)^{8\cdot 2} - 100\left(1+\frac{q}{2}\right)^{7.5\cdot 2} = 100\left(1+\frac{q}{2}\right)^{15}\left(\left(1+\frac{q}{2}\right)-1\right)$

$M: 200(1+8q) - 200(1+7.5q) = 200(0.5q)$

Eric deposits 100 into a savings account at time 0, which pays interest at a nominal rate of q, compounded semiannually.

Mike deposits 200 into a different savings account at time 0, which pays simple interest at an annual rate of q.

Eric and Mike earn the same amount of interest during the last 6 months of the 8th year.

Calculate q. [9.46%]

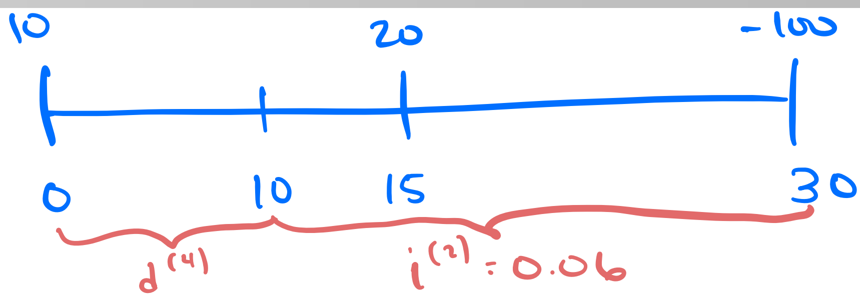
9.06%, 9.26%, 9.46%, 9.66%, 9.86%

$\cancel{100}\, \cancel{q} = \cancel{100}\left(1+\frac{q}{2}\right)^{15}\left(\frac{q}{2}\right)$

$2 = \left(1+\frac{q}{2}\right)^{15}$

$\left(2^{1/15}-1\right)2 = q$

SOA Examples Cont.



$t: 30$

$$100 = 20\left(1+\frac{0.06}{2}\right)^{30} + 10\left(1-\frac{d^{(4)}}{4}\right)^{-40}(1.03)^{40}$$

Jeff deposits 10 into a fund today and 20 fifteen years later. Interest is credited at a nominal discount rate of $d^{(4)}$ compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semiannually thereafter. The accumulated balance in the fund at the end of 30 years is 100.

$t: 15$

$$100(1.03)^{-2\cdot 15} = 20 + 10\left(1-\frac{d^{(4)}}{4}\right)^{-40}(1.03)^{10}$$

Calculate $d^{(4)}$. [4.53%]

4.33%, 4.43%, 4.53%, 4.63%, 4.73%

$t = 10$

$$100(1.03)^{-40} = 20(1.03)^{-10} + 10\left(1-\frac{d^{(4)}}{4}\right)^{-40}$$

$t = 0$

$$100(1.03)^{-40}\left(1-\frac{d^{(4)}}{4}\right)^{40} = 10 + 20(1.03)^{-10}\left(1-\frac{d^{(4)}}{4}\right)^{40}$$

$\delta_t = 0.001 + 0.02t$

You put in 100 at time 1.

Calculate value at time 4.

$$100\, e^{\int_1^4 \delta_t dt}$$

$$100\, e^{\int_1^4 0.001 + 0.02t dt}$$

$$100\, e^{\left[0.001t + 0.01t^2\right]_1^4}$$

116.53